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BEM analysis of fracture problems in three-dimensional thermoelasticity using *J*-integral

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Abstract

This paper describes the formulation and numerical implementation of the *J*-integral applied to three-dimensional thermoelasticity using a boundary element technique. The mixed-mode stress intensity factors has been evaluated through a decomposition technique. In this technique, the *J*-integral is split into J^{I}, J^{II} and J^{III}, associated to the three basic modes of fracture. The decomposition technique is compared to the crack opening displacement technique. Good accuracy was obtained in the mixed-mode test examples. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Since its introduction as a fracture mechanics parameter, the stress intensity factor has gained wide acceptance as a key parameter in the determination of crack behaviour in LEFM. Several techniques can be applied in the calculation of stress intensity factors, in two and three dimensions, under thermo-mechanical loads. One of these techniques is the crack opening displacement (COD) criterion, based on the extrapolation of the displacement field in the vicinity of the crack front. Although this technique is easy to implement, its main drawback is that a high level of mesh refinement is required to obtain accurate results. Another prominent technique, for the characterisation of cracks, is the use of path independent integrals based on conservation laws. The most widely known of these integrals is the *J*-integral of Rice (1968). Since the introduction of the *J*-integral, some researchers have attempted to generalise this parameter to characterise singularities subjected to other kind of loads, such as inertia effects or thermal gradients. A number of integrals have been developed which allows its use in thermoelastic fracture mechanics problems (Ainsworth et al., 1978; Wilson and Yu, 1979; Kishimoto et al., 1980).

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The implementation of the J-integral in BEM was presented by Aliabadi (1990) for two-dimensional elasticity and by Rigby and Aliabadi (1993) and Huber et al. (1993) for three-dimensional elasticity. Later Rigby and Aliabadi (1998) presented a correct decomposition for mixed-mode problems. Recently, application of the J-integral to two-dimensional thermoelasticity was presented by Prasad et al. (1994, 1996) including steady state and time-dependant problems.

In this paper, the derivation of the J-integral for thermoelasticity will be presented. The integral, resulting in a contour integral plus two area integrals, are decomposed using the symmetric and antisymmetric thermoelastic fields. This decomposition allows mode I, II and III stress intensity factors to be assessed as the integral is converted into the sum of three integrals $(J^{\mathrm{I}}, J^{\mathrm{II}}, J^{\mathrm{III}})$ associated to the three modes of fracture. A method of calculating the values at internal points is presented and the kernels arising from these new equations are listed. In the case of contour's end points, where the point belongs to an element, shape function differentiation is used in the evaluation of the required values. The implementation of the integration in the three-dimensional DBEM is also presented. Finally, several examples are presented to illustrate the accuracy and efficiency of the proposed technique.

2. The J-integral for thermoelasticity

Rice's J-integral derives from Eshelby's momentum tensor, which is

$$P_{ij} = W\delta_{ij} - \sigma_{ij}\frac{\partial u_i^{\mathrm{e}}}{\partial x_k}$$

as was denoted by Amestoy et al. (1981) in which W is the strain energy density, σ_{ij} is the stress tensor and u_i^{e} is the elastic displacement field. All the parameters in this tensor are in terms of the crack front coordinate system illustrated in Fig. 1.

Considering a generic contour C enclosing an area Ω (Fig. 1) defined in the plane $x_3 = 0$ and taking into account the following property of P_{ij} (Rigby and Aliabadi, 1998):

$$P_{ij,j} = 0,$$

the integral of $P_{ij,j}$ over any area Ω, excluding the crack singularity can be presented as

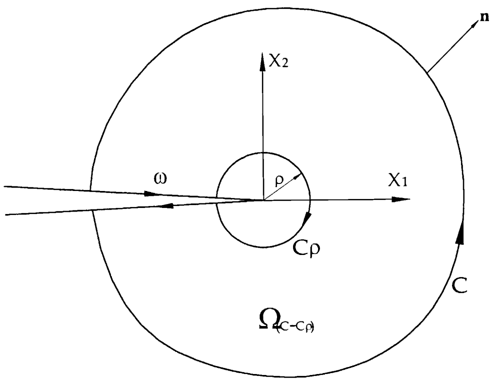

Fig. 1. Contour perpendicular to crack front.

$$\int_{\Omega(C-C_\rho)} \frac{\partial}{\partial x_j}\left(W\delta_{kj} - \sigma_{ij}\frac{\partial u_i^{\mathrm{e}}}{\partial x_k}\right)\mathrm{d}\Omega = 0, \tag{1}$$

where $\Omega(C - C_\rho) = \Omega(C) - \Omega(C_\rho)$ denotes the area delimited by the contours C, C_ρ and crack surface ω. Under thermal strains, the elastic strain tensor $\epsilon_{ij}^{\mathrm{e}}$ is defined as

$$\epsilon_{ij}^{\mathrm{e}} = \epsilon_{ij} - \epsilon_{ij}^{\theta}, \tag{2}$$

where ϵ_{ij} and ϵ_{ij}^{θ} are the total and thermal strains, respectively. Substituting Eq. (2) into Eq. (1) yields

$$\int_{\Omega(C-C_\rho)} \frac{\partial}{\partial x_j}\left(W\delta_{kj} - \sigma_{ij}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega + \int_{\Omega(C-C_\rho)} \sigma_{ij}\frac{\partial \epsilon_{ij}^{\theta}}{\partial x_k}\,\mathrm{d}\Omega = 0, \tag{3}$$

where the thermal strains has no influence on the strain energy density W. Applying Green's theorem, which is

$$\int_{\Omega}\left(\frac{\partial Q}{\partial x_1} - \frac{\partial T}{\partial x_2}\right)\mathrm{d}\Omega = \int_{\Gamma}(T\,\mathrm{d}x_1 + Q\,\mathrm{d}x_2),$$

and since $\mathrm{d}x_1 = -n_2\,\mathrm{d}\Gamma$ and $\mathrm{d}x_2 = n_1\,\mathrm{d}\Gamma$ (being $\mathbf{n}$ the normal to the contour Γ), the following can be obtained from Eq. (3):

$$\int_{\Gamma}\left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C-C_\rho)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega + \int_{\Omega(C-C_\rho)}\sigma_{ij}\frac{\partial \epsilon_{ij}^{\theta}}{\partial x_k}\,\mathrm{d}\Omega = 0, \tag{4}$$

where $\Gamma = C + C_\rho + \omega$ represents the contour around the area $\Omega(C - C_\rho)$. Eq. (4) can be reordered as

$$\begin{aligned}&\int_{C+\omega}\left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega + \int_{\Omega(C)}\sigma_{ij}\frac{\partial \epsilon_{ij}^{\theta}}{\partial x_k}\,\mathrm{d}\Omega \\ &\quad = -\int_{C_\rho}\left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C_\rho)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega - \int_{\Omega(C_\rho)}\sigma_{ij}\frac{\partial \epsilon_{ij}^{\theta}}{\partial x_k}\,\mathrm{d}\Omega.\end{aligned} \tag{5}$$

Taking the contour C_ρ as a circular contour, being ρ the radius of the contour, the area terms in the right-hand side of Eq. (5) will vanish as $\rho \to 0$. The J-integral J_k is defined from Eq. (5) as

$$\begin{aligned}J_k &= \int_{\Gamma_\rho}\left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma = \int_{C+\omega}\left(Wn_k - \sigma_{ij}\frac{\partial u_i}{\partial x_k}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_k}\right)\mathrm{d}\Omega \\ &\quad + \int_{\Omega(C)}\sigma_{ij}\frac{\partial \epsilon_{ij}^{\theta}}{\partial x_k}\,\mathrm{d}\Omega,\end{aligned} \tag{6}$$

where Γ_ρ is a contour identical to C_ρ but proceeding in an anticlockwise direction. The integral J_k is defined in the plane $x_3 = 0$ for any position on the crack front. Considering a traction free crack and taking $k = 1$, the contour integral over the crack faces ω is zero. Accordingly, Eq. (6) becomes

$$\begin{aligned}J_1 &= \int_{\Gamma_\rho}\left(Wn_1 - \sigma_{ij}\frac{\partial u_i}{\partial x_1}n_j\right)\mathrm{d}\Gamma \\ &= \int_{C}\left(Wn_1 - \sigma_{ij}\frac{\partial u_i}{\partial x_1}n_j\right)\mathrm{d}\Gamma - \int_{\Omega(C)}\frac{\partial}{\partial x_3}\left(\sigma_{i3}\frac{\partial u_i}{\partial x_1}\right)\mathrm{d}\Omega + \int_{\Omega(C)}\sigma_{ij}\alpha\frac{\partial \theta}{\partial x_1}\delta_{ij}\,\mathrm{d}\Omega,\end{aligned} \tag{7}$$

where the relation $\epsilon_{ij}^{\theta} = \alpha\theta\delta_{ij}$ has been employed. Eq. (7) is the three-dimensional version, in absence of inertia effects and body forces, of the $\hat{J}_1$ integral presented by Kishimoto et al. (1980). The path-independence of J_1 can be demonstrated from previous equation. Considering the contour Γ_ρ held constant, the right-hand side of Eq. (7) is constant for any contour C, i.e. the right-hand side is path-area independent.

3. Mixed mode *J*-integral

The integral J_k is related with the three modes of fracture through the integrals J^{I}, J^{II} and J^{III} as follows (Cherepanov, 1979):

$$J_1 = J^{\mathrm{I}} + J^{\mathrm{II}} + J^{\mathrm{III}}, \tag{8}$$

$$J_2 = -2\sqrt{J^{\mathrm{I}} J^{\mathrm{II}}}. \tag{9}$$

Nevertheless, the use of J_2 will involve the integration of singular fields over the crack surface. In addition, Herrmann and Herrmann (1981) have demonstrated that J_2 is path independent only if the integral of Wn_2 over the crack faces vanishes.

Rigby and Aliabadi (1998) presented another approach, known as the decomposition method, from which the integrals J^{I}, J^{II} and J^{III} in elasticity can be obtained directly from J_1, avoiding the use of J_2. A similar approach can be applied for decoupling the integral J_1 in Eq. (7), where the effect of thermal gradients are also considered.

First, the integral J_1 is splitted into two parts:

$$J_1 = J^{\mathrm{S}} + J^{\mathrm{AS}}, \tag{10}$$

where J^{S} and J^{AS} are found from the symmetric and antisymmetric thermoelastic fields about the crack plane, respectively. As the mode I thermoelastic fields are symmetric to the crack plane, the following relationship holds:

$$J^{\mathrm{S}} = J^{\mathrm{I}}, \quad J^{\mathrm{AS}} = J^{\mathrm{II}} + J^{\mathrm{III}}.$$

The integrals J^{II} and J^{III} can be obtained from J^{AS} by making additional analysis on the antisymmetric fields. Once obtained the integral J_1 as separated contributions of mode I, II and III *J*-integrals, the stress intensity factors can be calculated as follows:

$$\begin{aligned} J_1 &= J^{\mathrm{I}} + J^{\mathrm{II}} + J^{\mathrm{III}} \\ &= \frac{1}{E^*}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right) + \frac{1}{2\mu}K_{\mathrm{III}}^2 \end{aligned} \tag{11}$$

being $E^* = E$ for plane stress and $E^* = E/(1-\nu^2)$ for plane strain.

4. Symmetric and antisymmetric components

The symmetric and antisymmetric components of the *J*-integral in Eq. (10), from the thermoelastic fields, will be obtained in this section.

These fields can be obtained by considering two points $P(x_1, x_2, x_3)$ and $P'(x_1, -x_2, x_3)$, which are placed symmetric with respect to the crack plane $x_2 = 0$, as illustrated in Fig. 2.

The symmetric and antisymmetric components of the temperature field can be obtained from the temperatures at points P and P' as

$$\theta^{\mathrm{S}} = \tfrac{1}{2}(\theta_P + \theta_{P'}), \tag{12}$$

$$\theta^{\mathrm{AS}} = \tfrac{1}{2}(\theta_P - \theta_{P'}) \tag{13}$$

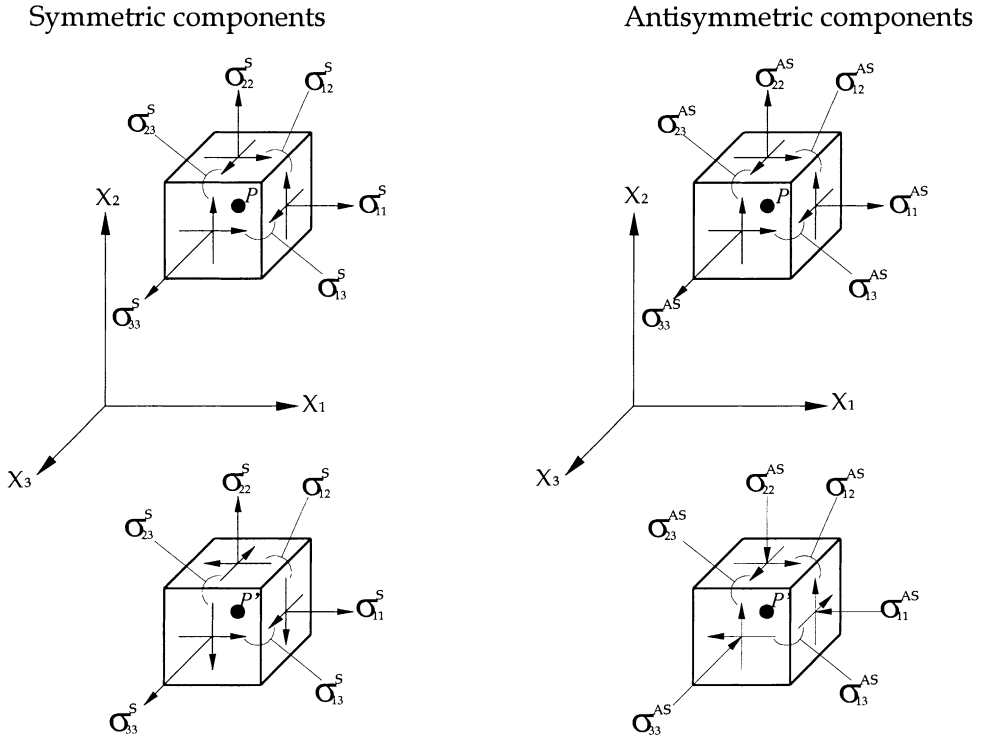

Fig. 2. Symmetric and antisymmetric components of stress at points P and P'.

being S and AS the symmetric and antisymmetric components, respectively, and where θ_P and $\theta_{P'}$ are the temperatures at points P and P'. Also, the temperature derivatives, can be obtained in the symmetric and antisymmetric components from Eqs. (12) and (13) as

$$\frac{\partial\theta^{\mathrm{S}}}{\partial x_j} = \frac{1}{2}\left\{\begin{matrix} \frac{\partial\theta_P}{\partial x_1} + \frac{\partial\theta_{P'}}{\partial x_1} \\ \frac{\partial\theta_P}{\partial x_2} - \frac{\partial\theta_{P'}}{\partial x_2} \\ \frac{\partial\theta_P}{\partial x_3} + \frac{\partial\theta_{P'}}{\partial x_3} \end{matrix}\right\}, \tag{14}$$

$$\frac{\partial\theta^{\mathrm{AS}}}{\partial x_j} = \frac{1}{2}\left\{\begin{matrix} \frac{\partial\theta_P}{\partial x_1} - \frac{\partial\theta_{P'}}{\partial x_1} \\ \frac{\partial\theta_P}{\partial x_2} + \frac{\partial\theta_{P'}}{\partial x_2} \\ \frac{\partial\theta_P}{\partial x_3} - \frac{\partial\theta_{P'}}{\partial x_3} \end{matrix}\right\}. \tag{15}$$

The stresses at points P and P' can be expressed in terms of the symmetric and antisymmetric components as

$$\left\{\begin{matrix} \sigma_{11P} \\ \sigma_{12P} \\ \sigma_{13P} \\ \sigma_{22P} \\ \sigma_{23P} \\ \sigma_{33P} \end{matrix}\right\} = \left\{\begin{matrix} \sigma^{\mathrm{S}}_{11P} \\ \sigma^{\mathrm{S}}_{12P} \\ \sigma^{\mathrm{S}}_{13P} \\ \sigma^{\mathrm{S}}_{22P} \\ \sigma^{\mathrm{S}}_{23P} \\ \sigma^{\mathrm{S}}_{33P} \end{matrix}\right\} + \left\{\begin{matrix} \sigma^{\mathrm{AS}}_{11P} \\ \sigma^{\mathrm{AS}}_{12P} \\ \sigma^{\mathrm{AS}}_{13P} \\ \sigma^{\mathrm{AS}}_{22P} \\ \sigma^{\mathrm{AS}}_{23P} \\ \sigma^{\mathrm{AS}}_{33P} \end{matrix}\right\}, \tag{16}$$

$$\begin{Bmatrix} \sigma_{11P'} \\ \sigma_{12P'} \\ \sigma_{13P'} \\ \sigma_{22P'} \\ \sigma_{23P'} \\ \sigma_{33P'} \end{Bmatrix} = \begin{Bmatrix} \sigma^{S}_{11P'} \\ -\sigma^{S}_{12P'} \\ \sigma^{S}_{13P'} \\ \sigma^{S}_{22P'} \\ -\sigma^{S}_{23P'} \\ \sigma^{S}_{33P'} \end{Bmatrix} + \begin{Bmatrix} -\sigma^{AS}_{11P'} \\ \sigma^{AS}_{12P'} \\ -\sigma^{AS}_{13P'} \\ -\sigma^{AS}_{22P'} \\ \sigma^{AS}_{23P'} \\ -\sigma^{AS}_{33P'} \end{Bmatrix}. \tag{17}$$

The symmetric and antisymmetric stress fields can be found by combining the stresses at points P and P' as follows:

$$\begin{Bmatrix} \sigma^{S}_{11} \\ \sigma^{S}_{12} \\ \sigma^{S}_{13} \\ \sigma^{S}_{22} \\ \sigma^{S}_{23} \\ \sigma^{S}_{33} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \sigma_{11P} + \sigma_{11P'} \\ \sigma_{12P} - \sigma_{12P'} \\ \sigma_{13P} + \sigma_{13P'} \\ \sigma_{22P} + \sigma_{22P'} \\ \sigma_{23P} - \sigma_{23P'} \\ \sigma_{33P} + \sigma_{33P'} \end{Bmatrix}, \tag{18}$$

$$\begin{Bmatrix} \sigma^{AS}_{11} \\ \sigma^{AS}_{12} \\ \sigma^{AS}_{13} \\ \sigma^{AS}_{22} \\ \sigma^{AS}_{23} \\ \sigma^{AS}_{33} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \sigma_{11P} - \sigma_{11P'} \\ \sigma_{12P} + \sigma_{12P'} \\ \sigma_{13P} - \sigma_{13P'} \\ \sigma_{22P} - \sigma_{22P'} \\ \sigma_{23P} + \sigma_{23P'} \\ \sigma_{33P} - \sigma_{33P'} \end{Bmatrix}. \tag{19}$$

The strains can be represented by the sum of its symmetric and antisymmetric components as

$$\epsilon_{ij} = \epsilon^{S}_{ij} + \epsilon^{AS}_{ij} = \frac{1}{2}\begin{Bmatrix} \epsilon_{11P} + \epsilon_{11P'} \\ \epsilon_{12P} - \epsilon_{12P'} \\ \epsilon_{13P} + \epsilon_{13P'} \\ \epsilon_{22P} + \epsilon_{22P'} \\ \epsilon_{23P} - \epsilon_{23P'} \\ \epsilon_{33P} + \epsilon_{33P'} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \epsilon_{11P} - \epsilon_{11P'} \\ \epsilon_{12P} + \epsilon_{12P'} \\ \epsilon_{13P} - \epsilon_{13P'} \\ \epsilon_{22P} - \epsilon_{22P'} \\ \epsilon_{23P} + \epsilon_{23P'} \\ \epsilon_{33P} - \epsilon_{33P'} \end{Bmatrix}. \tag{20}$$

The symmetric and antisymmetric components of the strain tensor are related to the displacements derivatives as

$$\epsilon^{\alpha}_{ij} = \frac{1}{2}\left(\frac{\partial u^{\alpha}_i}{\partial x_j} + \frac{\partial u^{\alpha}_j}{\partial x_i}\right), \tag{21}$$

where $\alpha = \mathrm{S}$ or AS. From now, on the sub-indices P and P' will be eliminated for simplicity, so f' will represent a field evaluated at P' and f will represent a field evaluated at P. The displacement derivatives can be obtained (Rigby and Aliabadi, 1998) from Eq. (21) as

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} &= \frac{\partial u^{S}_i}{\partial x_j} + \frac{\partial u^{AS}_i}{\partial x_j} \\ &= \frac{1}{2}\begin{Bmatrix} \frac{\partial u_1}{\partial x_j} + \frac{\partial u'_1}{\partial x_j} \\ \frac{\partial u_2}{\partial x_j} - \frac{\partial u'_2}{\partial x_j} \\ \frac{\partial u_3}{\partial x_j} + \frac{\partial u'_3}{\partial x_j} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \frac{\partial u_1}{\partial x_j} - \frac{\partial u'_1}{\partial x_j} \\ \frac{\partial u_2}{\partial x_j} + \frac{\partial u'_2}{\partial x_j} \\ \frac{\partial u_3}{\partial x_j} - \frac{\partial u'_3}{\partial x_j} \end{Bmatrix}. \end{aligned} \tag{22}$$

Eq. (7) can be written using the symmetric and antisymmetric components of the fields derived in Eqs. (14)–(22) as

$$J_1 = \int_C \left\{ \left[\int_0^{\epsilon_{ij}} \left(\sigma_{ij}^{\mathrm{S}} + \sigma_{ij}^{\mathrm{AS}}\right) \mathrm{d}\left(\epsilon_{ij}^{\mathrm{S}} + \epsilon_{ij}^{\mathrm{AS}}\right) \right] n_1 - \left(\sigma_{ij}^{\mathrm{S}} + \sigma_{ij}^{\mathrm{AS}}\right) n_j \frac{\partial}{\partial x_1} \left(u_i^{\mathrm{S}} + u_i^{\mathrm{AS}}\right) \right\} \mathrm{d}\Gamma \\ - \int_{\Omega(C)} \frac{\partial}{\partial x_3} \left[\left(\sigma_{i3}^{\mathrm{S}} + \sigma_{i3}^{\mathrm{AS}}\right) \frac{\partial}{\partial x_1} \left(u_i^{\mathrm{S}} + u_i^{\mathrm{AS}}\right) \right] \mathrm{d}\Omega + \int_{\Omega(C)} \alpha \left(\sigma_{ij}^{\mathrm{S}} + \sigma_{ij}^{\mathrm{AS}}\right) \frac{\partial}{\partial x_1} \left(\theta^{\mathrm{S}} + \theta^{\mathrm{AS}}\right) \delta_{ij} \, \mathrm{d}\Omega, \tag{23}$$

where the definition of $W = \int_0^{\epsilon_{ij}} \sigma_{ij} \, \mathrm{d}\epsilon_{ij}$ has been used.

Considering a contour C symmetric about the crack plane $x_2 = 0$, we can find that for any pair of symmetric points P and P', the following relation holds for the normals $\mathbf{n}$ and $\mathbf{n}'$:

$$\mathbf{n} = (n_1, n_2, 0), \quad \mathbf{n}' = (n_1, -n_2, 0),$$

and for the integrands in Eq. (23),

$$\sigma_{ij}^{\prime\alpha} \, \mathrm{d}\epsilon_{ij}^{\prime\beta} = \pm \sigma_{ij}^{\alpha} \, \mathrm{d}\epsilon_{ij}^{\beta}, \tag{24}$$

$$\sigma_{ij}^{\prime\alpha} n_j' \frac{\partial u_i^{\prime\beta}}{\partial x_1} = \pm \sigma_{ij}^{\alpha} n_j \frac{\partial u_i^{\beta}}{\partial x_1}, \tag{25}$$

$$\sigma_{i3}^{\prime\alpha} \frac{\partial u_i^{\prime\beta}}{\partial x_1} = \pm \sigma_{i3}^{\alpha} \frac{\partial u_i^{\beta}}{\partial x_1}, \tag{26}$$

$$\sigma_{ii}^{\prime\alpha} \frac{\partial \theta^{\prime\beta}}{\partial x_1} = \pm \sigma_{ii}^{\alpha} \frac{\partial \theta^{\beta}}{\partial x_1}. \tag{27}$$

being α, $\beta = $ S or AS. The positive sign in the right-hand side of Eqs. (24)–(27) represent the case $\alpha = \beta$ and the negative sign the case $\alpha \neq \beta$. As the contour C is taken symmetric about the crack plane, the integrals in Eq. (23) cancel each other at symmetric points for the case $\alpha \neq \beta$. Thus, Eq. (23) becomes

$$J_1 = \sum_{\alpha=1}^{2} \int_C \left(W^{\alpha} n_1 - \sigma_{ij}^{\alpha} \frac{\partial u_i^{\alpha}}{\partial x_1} n_j \right) \mathrm{d}\Gamma - \int_{\Omega(C)} \frac{\partial}{\partial x_3} \left(\sigma_{i3}^{\alpha} \frac{\partial u_i^{\alpha}}{\partial x_1} \right) \mathrm{d}\Omega + \int_{\Omega(C)} \sigma_{ii}^{\alpha} \alpha \frac{\partial \theta^{\alpha}}{\partial x_1} \, \mathrm{d}\Omega = J^{\mathrm{S}} + J^{\mathrm{AS}}, \tag{28}$$

where $\alpha = 1$, 2 denotes symmetric, S, and antisymmetric, AS, components, respectively. As was stated before, mode I correspond to the integral J^{S} while the integral J^{AS} is related to modes II and III. The integral J_1 will be decoupled into mode I, II and III terms in Section 5.

5. Decomposition of integrands

In this section, the integrands of integral J_1 will be decomposed into their mode I, II and III components. Being the symmetric components representative of mode I, the decomposition method will involve a further decoupling of the antisymmetric components into modes II and III.

The decomposition of stresses represented by

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{I}} + \sigma_{ij}^{\mathrm{II}} + \sigma_{ij}^{\mathrm{III}}$$

has been given by many authors (Nikishkov and Atluri, 1987; Shivakumar and Raju, 1990; Rigby and Aliabadi, 1993; Huber et al., 1993). Recently, Rigby and Aliabadi (1998) gave the proper decomposition of stresses, since they showed that the expressions used in previous papers were incorrect. So, the correct decomposition of stresses is as follows:

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{I}} + \sigma_{ij}^{\mathrm{II}} + \sigma_{ij}^{\mathrm{III}} = \frac{1}{2}\left\{\begin{matrix} \sigma_{11}+\sigma'_{11} \\ \sigma_{12}-\sigma'_{12} \\ \sigma_{13}+\sigma'_{13} \\ \sigma_{22}+\sigma'_{22} \\ \sigma_{23}-\sigma'_{23} \\ \sigma_{33}+\sigma'_{33} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} \sigma_{11}-\sigma'_{11} \\ \sigma_{12}+\sigma'_{12} \\ 0 \\ \sigma_{22}-\sigma'_{22} \\ 0 \\ \sigma_{33}-\sigma'_{33} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} 0 \\ 0 \\ \sigma_{13}-\sigma'_{13} \\ 0 \\ \sigma_{23}+\sigma'_{23} \\ 0 \end{matrix}\right\}. \tag{29}$$

The strain decomposition is derived from stress and temperature decomposition by the application of Hooke's law for thermoelasticity, which is

$$\epsilon_{ij} = \frac{1+v}{E}\sigma_{ij} - \frac{v}{E}\sigma_{kk}\delta_{ij} + \alpha\theta\delta_{ij} \tag{30}$$

by combining Eqs. (13) and (29) into Eq. (30), e.g.,

$$\epsilon_{11}^{\mathrm{II}} = \frac{1+v}{2E}\left(\sigma_{ij} - \sigma'_{ij}\right) - \frac{v}{E}\left(\sigma_{kk} - \sigma'_{kk}\right) + \alpha\left(\theta - \theta'\right) = \frac{1}{2}\left(\epsilon_{11} - \epsilon'_{11}\right). \tag{31}$$

This leads to

$$\epsilon_{ij} = \epsilon_{ij}^{\mathrm{I}} + \epsilon_{ij}^{\mathrm{II}} + \epsilon_{ij}^{\mathrm{III}} = \frac{1}{2}\left\{\begin{matrix} \epsilon_{11}+\epsilon'_{11} \\ \epsilon_{12}-\epsilon'_{12} \\ \epsilon_{13}+\epsilon'_{13} \\ \epsilon_{22}+\epsilon'_{22} \\ \epsilon_{23}-\epsilon'_{23} \\ \epsilon_{33}+\epsilon'_{33} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} \epsilon_{11}-\epsilon'_{11} \\ \epsilon_{12}+\epsilon'_{12} \\ 0 \\ \epsilon_{22}-\epsilon'_{22} \\ 0 \\ \epsilon_{33}-\epsilon'_{33} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} 0 \\ 0 \\ \epsilon_{13}-\epsilon'_{13} \\ 0 \\ \epsilon_{23}+\epsilon'_{23} \\ 0 \end{matrix}\right\}. \tag{32}$$

The mode I, II and III displacement derivatives can be derived from Eq. (32) by using the relation between displacements and strains as

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i^{\mathrm{I}}}{\partial x_j} + \frac{\partial u_i^{\mathrm{II}}}{\partial x_j} + \frac{\partial u_i^{\mathrm{III}}}{\partial x_j}, \tag{33}$$

where

$$\frac{\partial u_i}{\partial x_1} = \frac{1}{2}\left\{\begin{matrix} \frac{\partial u_1}{\partial x_1}+\frac{\partial u'_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1}-\frac{\partial u'_2}{\partial x_1} \\ \frac{\partial u_3}{\partial x_1}+\frac{\partial u'_3}{\partial x_1} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} \frac{\partial u_1}{\partial x_1}-\frac{\partial u'_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1}+\frac{\partial u'_2}{\partial x_1} \\ 0 \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} 0 \\ 0 \\ \frac{\partial u_3}{\partial x_1}-\frac{\partial u'_3}{\partial x_1} \end{matrix}\right\}, \tag{34}$$

$$\frac{\partial u_i}{\partial x_2} = \frac{1}{2}\left\{\begin{matrix} \frac{\partial u_1}{\partial x_2}+\frac{\partial u'_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_2}-\frac{\partial u'_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_2}+\frac{\partial u'_3}{\partial x_2} \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} \frac{\partial u_1}{\partial x_2}-\frac{\partial u'_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_2}+\frac{\partial u'_2}{\partial x_2} \\ 0 \end{matrix}\right\} + \frac{1}{2}\left\{\begin{matrix} 0 \\ 0 \\ \frac{\partial u_3}{\partial x_2}-\frac{\partial u'_3}{\partial x_2} \end{matrix}\right\}, \tag{35}$$

$$\frac{\partial u_i}{\partial x_3} = \frac{1}{2}\begin{Bmatrix} \frac{\partial u_1}{\partial x_3} + \frac{\partial u_1'}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} - \frac{\partial u_2'}{\partial x_3} \\ \frac{\partial u_3}{\partial x_3} + \frac{\partial u_3'}{\partial x_3} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} 0 \\ 0 \\ \frac{\partial u_3}{\partial x_3} - \frac{\partial u_3'}{\partial x_3} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} \frac{\partial u_1}{\partial x_3} - \frac{\partial u_1'}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} + \frac{\partial u_2'}{\partial x_3} \\ 0 \end{Bmatrix}. \tag{36}$$

The decomposition of the antisymmetric components of temperatures can be achieved from Eqs. (27) and (29). Since the antisymmetric component of σ_{kk} only affects mode II (Eq. (29)) and by virtue of Eq. (27), the temperature decomposition can be written as

$$\theta = \theta^{\mathrm{I}} + \theta^{\mathrm{II}} \tag{37}$$

being $\theta^{\mathrm{I}} = \theta^{\mathrm{S}}$, $\theta^{\mathrm{II}} = \theta^{\mathrm{AS}}$ and $\theta^{\mathrm{III}} = 0$. The temperature derivative can be decomposed from Eqs. (12), (13) and (37) as

$$\frac{\partial \theta}{\partial x_1} = \frac{\partial \theta^{\mathrm{I}}}{\partial x_1} + \frac{\partial \theta^{\mathrm{II}}}{\partial x_1} + \frac{\partial \theta^{\mathrm{III}}}{\partial x_1} = \frac{1}{2}\left(\frac{\partial \theta}{\partial x_1} + \frac{\partial \theta'}{\partial x_1}\right) + \frac{1}{2}\left(\frac{\partial \theta}{\partial x_1} - \frac{\partial \theta'}{\partial x_1}\right) + 0. \tag{38}$$

Returning to the J-integral in Eq. (28), the integrand on the first domain integral can be replaced by

$$\frac{\partial}{\partial x_3}\left(\sigma_{i3}^{\alpha}\frac{\partial u_i^{\alpha}}{\partial x_1}\right) = \frac{\partial \sigma_{i3}^{\alpha}}{\partial x_3}\frac{\partial u_i^{\alpha}}{\partial x_1} + \sigma_{i3}^{\alpha}\frac{\partial^2 u_i^{\alpha}}{\partial x_1 \partial x_3} \tag{39}$$

for $\alpha = \mathrm{S}$ or AS, and since the symmetric and antisymmetric fields exhibit equilibrium of forces (Rigby and Aliabadi, 1998), the following holds:

$$\frac{\partial \sigma_{i3}^{\alpha}}{\partial x_3} = -\left(\frac{\partial \sigma_{i1}^{\alpha}}{\partial x_1} + \frac{\partial \sigma_{i2}^{\alpha}}{\partial x_2}\right). \tag{40}$$

By replacing Eqs. (39) and (40) into Eq. (28) and by applying the thermoelastic fields decoupled in Eqs. (29)–(38), the mode I, II and III J-integral, which allows the characterisation of cracks under thermomechanical loads, can be written as

$$J^{\alpha} = \int_C \left(W^{\alpha} n_1 - \sigma_{ij}^{\alpha}\frac{\partial u_i^{\alpha}}{\partial x_1} n_j\right) \mathrm{d}\Gamma + \int_{\Omega(C)} \left(\frac{\partial \sigma_{i1}^{\alpha}}{\partial x_1} + \frac{\partial \sigma_{i2}^{\alpha}}{\partial x_2}\right)\frac{\partial u_i^{\alpha}}{\partial x_1}\,\mathrm{d}\Omega - \int_{\Omega(C)} \sigma_{i3}^{\alpha}\frac{\partial^2 u_i^{\alpha}}{\partial x_1 \partial x_3}\,\mathrm{d}\Omega + \int_{\Omega(C)} \sigma_{ii}^{\alpha}\alpha\frac{\partial \theta^{\alpha}}{\partial x_1}\,\mathrm{d}\Omega, \tag{41}$$

where $\alpha = $ I, II, III. In the absence of thermal gradients, the integral in Eq. (41) reduces to the integral quoted by Rigby and Aliabadi (1998) for three-dimensional elasticity. The stress intensity factors can be obtained from the above integral using Eq. (11).

6. Thermoelastic fields at internal points for *J*-integral

The thermoelastic variables and their derivatives are required at internal points for the implementation of the J-integral in Eq. (41). All the kernels arising are listed in Appendix A.

The temperatures $\theta(\mathbf{X}')$ at an internal point $\mathbf{X}'$, necessary for the calculation of internal stresses in Eq. (41) (dell'Erba et al., 2000), is

$$\theta(\mathbf{X}') - \int_{\Gamma} q^*(\mathbf{X}', \mathbf{x})\theta(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = -\int_{\Gamma} \theta^*(\mathbf{X}', \mathbf{x})q(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \tag{42}$$

The temperature derivatives are also required for the calculation of the area integral and stress derivatives in Eq. (41). The temperature derivatives at internal points, can be obtained by deriving Eq. (42) with respect to the coordinates x_k, as

$$\theta_{,k}(\mathbf{X}') - \int_\Gamma q_k^{**}(\mathbf{X}',\mathbf{x})\theta(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) = -\int_\Gamma \theta_k^{**}(\mathbf{X}',\mathbf{x})q(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \tag{43}$$

The displacement derivatives $\partial u_i/\partial x_k$ at an interior point $\mathbf{X}'$ can be calculated from the displacement equation for interior points (dell'Erba et al., 2000) by differentiating with respect to the coordinates x_k. It can be written as follows:

$$\begin{aligned} u_{i,k}(\mathbf{X}') &+ \int_\Gamma T_{ij,k}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{P}_{i,k}(\mathbf{X}',\mathbf{x})\theta(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \\ &= \int_\Gamma U_{ij,k}(\mathbf{X}',\mathbf{x})t_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{Q}_{i,k}(\mathbf{X}',\mathbf{x})q(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \end{aligned} \tag{44}$$

The area integral in Eq. (41) also requires the second derivatives of displacements $\partial^2 u_i/\partial x_k \partial x_m$ which can be achieved by differentiating Eq. (44) with respect to the coordinates x_m, as

$$\begin{aligned} u_{i,km}(\mathbf{X}') &+ \int_\Gamma T_{ij,km}(\mathbf{X}',\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{P}_{i,km}(\mathbf{X}',\mathbf{x})\theta(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \\ &= \int_\Gamma U_{ij,km}(\mathbf{X}',\mathbf{x})t_j(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{Q}_{i,km}(\mathbf{X}',\mathbf{x})q(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}). \end{aligned} \tag{45}$$

The stress tensor $\sigma_{ij}(\mathbf{X}')$ at internal points $\mathbf{X}'$ can be written as (dell'Erba et al., 2000):

$$\begin{aligned} \sigma_{ij}(\mathbf{X}') &+ \int_\Gamma T_{kij}(\mathbf{X}',\mathbf{x})u_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{P}_{ij}(\mathbf{X}',\mathbf{x})\theta(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) + \frac{E}{(1-2v)}\alpha\theta(\mathbf{X}')\delta_{ij} \\ &= \int_\Gamma U_{kij}(\mathbf{X}',\mathbf{x})t_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{Q}_{ij}(\mathbf{X}',\mathbf{x})q(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}), \end{aligned} \tag{46}$$

where the temperature $\theta(\mathbf{X}')$ is obtained from Eq. (42).

For the calculation of the area integral in Eq. (41), the derivatives of stresses $\sigma_{ij,m}$ are required at internal points $\mathbf{X}'$. By differentiating Eq. (46) with respect to the coordinates x_m, the equation for stress derivatives at an internal point $\mathbf{X}'$ can be obtained as

$$\begin{aligned} \sigma_{ij,m}(\mathbf{X}') &+ \int_\Gamma T_{kij,m}(\mathbf{X}',\mathbf{x})u_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{P}_{ij,m}(\mathbf{X}',\mathbf{x})\theta(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) + \frac{E}{(1-2v)}\alpha\theta_{,m}(\mathbf{X}')\delta_{ij} \\ &= \int_\Gamma U_{kij,m}(\mathbf{X}',\mathbf{x})t_k(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma \overline{Q}_{ij,m}(\mathbf{X}',\mathbf{x})q(\mathbf{x})\,\mathrm{d}\Gamma(\mathbf{x}) \end{aligned} \tag{47}$$

being the temperature derivative $\theta_{,m}(\mathbf{X}')$ obtained from Eq. (43).

The J-integral in Eq. (41) requires all the internal values defined in terms of the local crack coordinate system shown in Fig. 3. All these quantities are calculated in the global coordinate system and then transformed into the local coordinate system using standard transformation equations for Cartesian vectors and tensors as

$$\theta_{,i} = a_{ij}\theta_{,j}, \quad u_{i,j} = a_{ik}a_{jl}u_{k,l}, \quad u_{i,jk} = a_{il}a_{jm}a_{kn}u_{l,mn}, \quad \sigma_{ij} = a_{ik}a_{jl}\sigma_{kl}, \quad \sigma_{ij,k} = a_{il}a_{jm}a_{kn}\sigma_{lm,n}, \tag{48}$$

where a_{ij} is the local transformation matrix given by the directional cosines of the unit vectors along the local axis X_1, X_2 and X_3 depicted in Fig. 3.

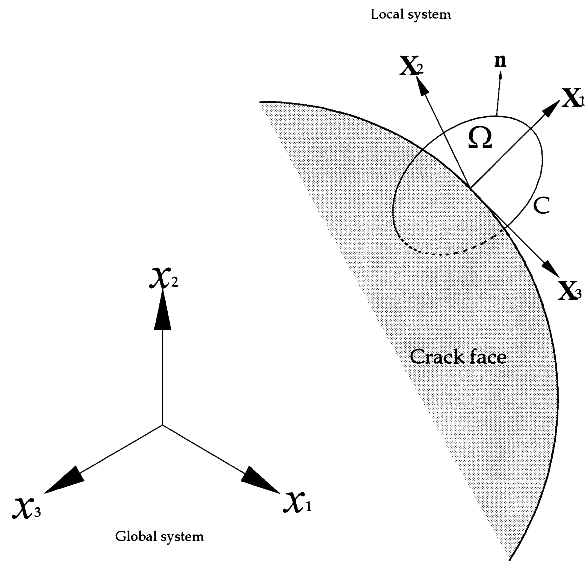

Fig. 3. Global coordinate system, local coordinate system and J contour.

7. Numerical implementation

The J-integral technique has been included as a post-processing technique, so it can be applied to the results of a particular model at a later stage to obtain stress intensity factors for different positions along the crack front. The necessary values at internal points for the calculation of the J-integral are calculated from Eqs. (42)–(47) by applying the boundary values $\theta(\mathbf{x})$, $q(\mathbf{x})$, $u_i(\mathbf{x})$ and $t_i(\mathbf{x})$ previously calculated at boundary points $\mathbf{x}$. The internal values, which are in the global coordinate system, are transformed into the local crack coordinate system for a particular position on the crack front by using Eq. (48). Finally, the fields associated to modes I, II and III are integrated separately to obtain integrals J^{I}, J^{II} and J^{III} from which the values of stress intensity factors are calculated.

In the next sections, the procedure adopted for calculating the contour and area integrals and for calculating the values over the crack faces (e.g. end points of the contour) will be explained in detail.

7.1. Contour and area integration

The procedure adopted in this work for the evaluation of contour and area integrals is similar to the one presented by Rigby and Aliabadi (1993) for the J-integral evaluation in three-dimensional elasticity in BEM.

The strategy can be described, with reference to Fig. 4, as follows:

(1) A contour of radius r, centred at a point O in the crack front, is placed perpendicular to the crack front. The area enclosed is divided in an even number of area segments.

(2) The internal points, where the internal values will be calculated, are located in concentric arcs of radius $r/3$, $2r/3$ and r in the fashion illustrated in Fig. 4.

(3) For internal points such as these belonging to the area segment highlighted in Fig. 4 ($A - J$) and their symmetric counterpart ($B' - J'$), the internal values required for the calculation of J-integral, are calculated by integrating Eqs. (42)–(47) around the surface of the body, including the crack faces.

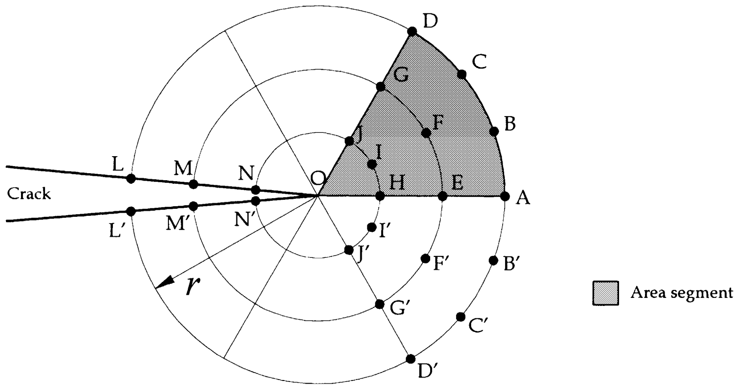

Fig. 4. Distribution of internal points for contour and area integration.

(4) For points such as $L - N$ and $L' - N'$, the required values are obtained by differentiating the shape function within the element. This will be explained in detail in Section 7.2.

(5) All the points are placed symmetrically with respect to the crack plane, e.g. the point B is symmetric to the point B'. By combining the results at symmetric points, the integrands for J^{I}, J^{II} and J^{III} can be obtained. Since all the integrands are symmetric with respect to the crack plane, the integration is carried out only for the top area and then multiplied by 2.

(6) The contribution to the contour integral from the segment highlighted in Fig. 4 is found by applying the four point Newton–Cotes formula to the contour integrands at points $A - D$.

(7) The contribution to the area integrals from this segment is obtained via line integrals. First, line integrals are calculated for the three arcs in the segment. A line integral L_1 is obtained by applying Simpson's rule to the area integrands at points $H - J$ in the inner arc. Simpson's rule is also applied to points $E - G$ in the middle arc to produce a line integral L_2. The line integral L_3 in the outer arc is calculated from the area integrands at points $A - D$ using the four point Newton–Cotes formula. Considering that the integral of the area integrands over an area $\Omega(\varepsilon)$ will tend to zero as $\varepsilon \to 0$, a line integral $L_0 = 0$ is assumed from the integration over an arc of $r = 0$ (point O). Thus, since the arcs are equally spaced, the total area integral contribution from the area segment can be calculated from the four line integrals L_0, L_1, L_2 and L_3 by using the four point Newton–Cotes formula, yielding

$$J_{\mathrm{A}} = \frac{r}{8}(L_0 + 3L_1 + 3L_2 + L_3) = \frac{r}{8}(3L_1 + 3L_2 + L_3),$$

which is equivalent to find the integral of the area integrands I_{A} by

$$J_{\mathrm{A}} = \int_{\theta_0}^{\theta_1} \int_0^r I_{\mathrm{A}} r \, \mathrm{d}r \, \mathrm{d}\theta.$$

Finally, the stress intensity factors are calculated from Eq. (11) as

$$K_{\mathrm{I}} = \sqrt{E^* J^{\mathrm{I}}}, \quad K_{\mathrm{II}} = \sqrt{E^* J^{\mathrm{II}}}, \quad K_{\mathrm{III}} = \sqrt{2\mu J^{\mathrm{III}}}.$$

7.2. Values at end points

As shown in Fig. 4, the path of the contour intersect the crack faces, e.g. at points L and L', where the values of the contour and area integrands are also required. Since the point is on the boundary, it will be within one of the boundary elements and, therefore, the calculation of the internal values by integrating

Eqs. (42)–(47) is singular when integrating in this element. One way to obtain these values is by differentiating the shape functions in the element which contains the point (Aliabadi and Rooke, 1991). Alternatively, these values can be calculated using Eqs. (42)–(47) and performing the singular integration within the element that contains the point. The former approach is used in this work (see Appendix B).

8. Elliptical crack in an infinite solid

The numerical example consist of an elliptical crack in an infinite solid. The geometry is illustrated in Fig. 5 and a plan view of the crack is included for which the ellipticity is 0.5.

Other geometrical relations were chosen as $a/R = 0.05$ and $h/R = 6.0$ in order to simulate the conditions of a crack in an infinite solid. The material parameters were taken as $E = 2.1 \times 10^5$ MPa, $\alpha = 1.65 \times 10^{-5}$/°C and $\nu = 1/3$ for all the examples and the results are independent of the thermal conductivity λ. Two different set of boundary conditions were applied and the results are compared with analytical solution given by Kassir and Sih (1967). For comparison, the solution obtained using the COD formulae, has also been included.

8.1. Case 1: symmetric boundary conditions

In this case, all the surfaces were maintained at constant temperature, i.e.,

$$\theta_0 = 100°\text{C on the crack surfaces,}$$
$$\theta_C = 0°\text{C on the remaining surfaces,}$$

and all the surfaces of the model are traction free. The stress intensity factor was normalised as $K_I^* = K_I/F$, where $F = 4\mu\alpha\theta_0\sqrt{b/\pi}$. The results for K_I^* as a function of the angle β (Fig. 5) compared with the analytical and the COD solution is shown in Fig. 6.

As can be seen from Fig. 6, there are good agreement between the numerical and analytical results.

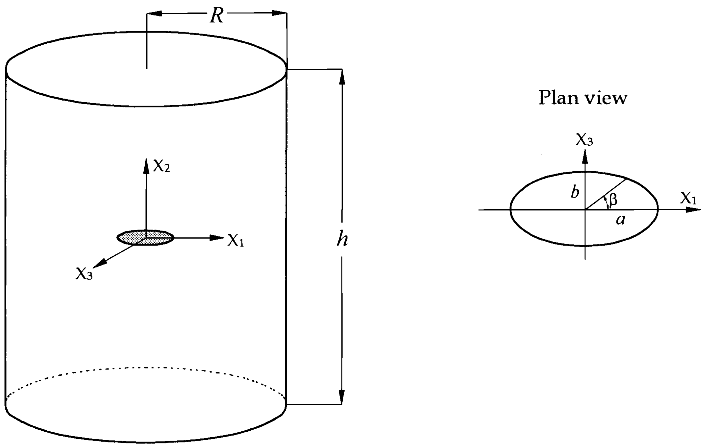

Fig. 5. Geometry of an elliptical crack in a circular cylinder.

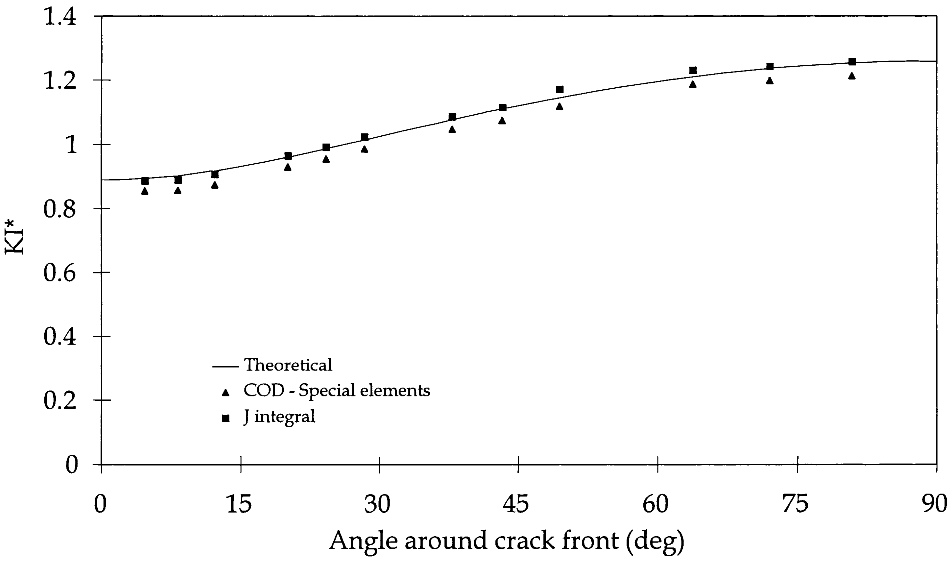

Fig. 6. Mode I stress intensity factor for an elliptical crack in an infinte domain.

8.2. Case 2: antisymmetric boundary conditions

In this case, the surfaces of the crack were insulated and a temperature gradient was set-up, perpendicular to the crack faces. The gradient arise from the temperature difference between the top and bottom surfaces of the cylinder. The boundary conditions are

$q_0 = 0$ on the crack surfaces and the external surface of the cylinder,

$\theta_0 = \pm 300°C$ on the top and bottom surfaces of the cylinder,

and all the surfaces of the model are traction free. For an elliptical crack, the antisymmetric boundary conditions generates a combination of K_{II} and K_{III} which occur simultaneously. The stress intensity factors are normalised as $K^*_{\mathrm{II}} = K_{\mathrm{II}}/F$ and $K^*_{\mathrm{III}} = K_{\mathrm{III}}/F$, where $F = 8\mu\alpha\theta_0 b^{3/2}/3h\sqrt{\pi}$. The results for K^*_{II} and K^*_{III} are presented in Figs. 7 and 8, respectively, and compared with the analytical and the COD solution.

9. Semi-elliptical surface crack

The geometry is illustrated in Fig. 9 for which the relation $a/b = 0.5$ was chosen. The crack was insulated and a uniform thermal gradient was set-up in the direction of the X axis by prescribing a temperature difference at the faces of the bar. The following boundary conditions were applied:

$\theta_1 = -100$ in the plane $X = 0$,

$\theta_2 = 100$ in the plane $X = T$,

$q = 0$ in the remaining surfaces,

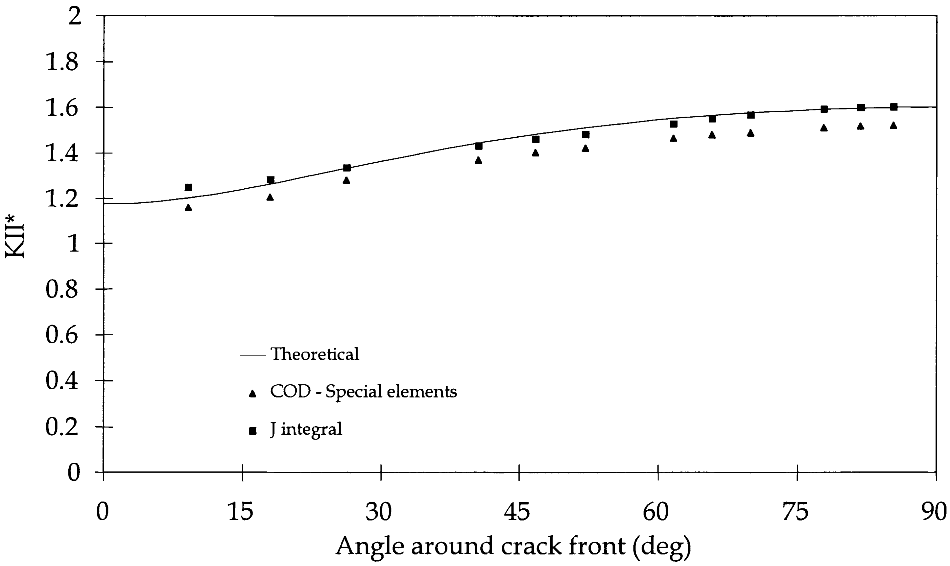

Fig. 7. Mode II stress intensity factor for an elliptical crack in an infinte domain.

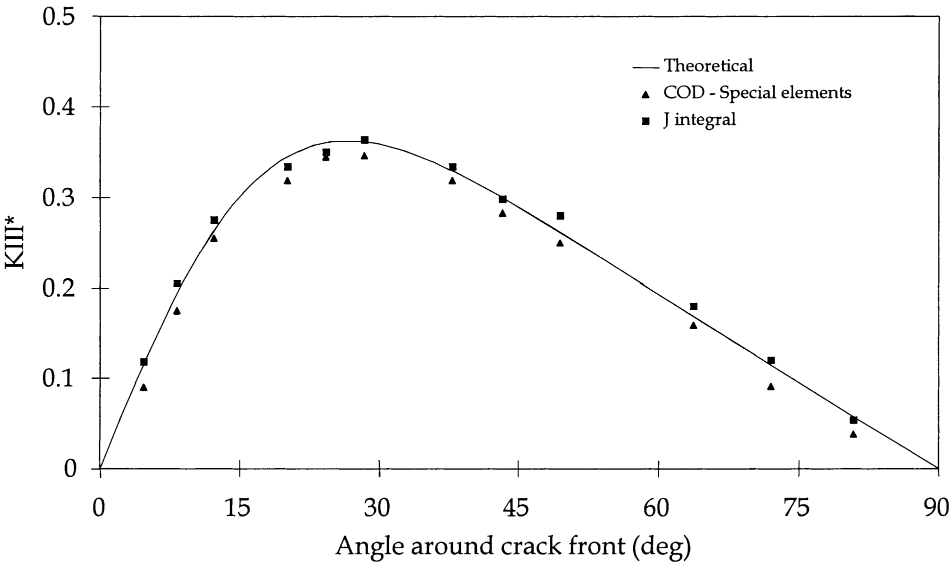

Fig. 8. Mode III stress intensity factor for an elliptical crack in an infinte domain.

and the top and bottom surfaces of the bar ($Y = \pm L$) were held against normal displacement. Results for normalised stress intensity factors are presented in Fig. 10, where $K^* = K/\alpha E T_0 \sqrt{W}$.

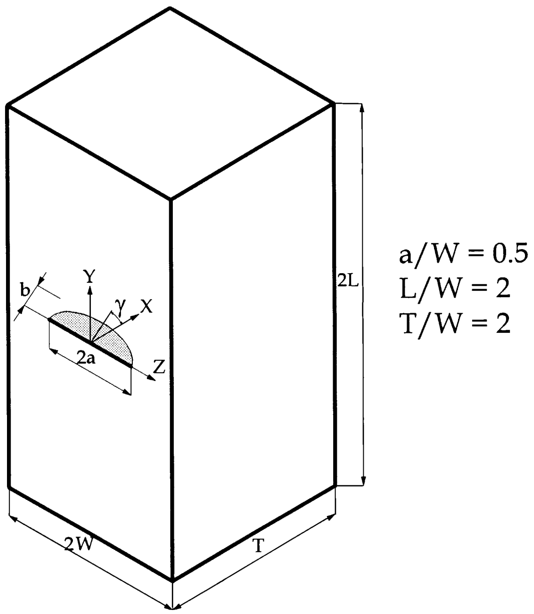

Fig. 9. Semi-elliptical surface crack in a bar.

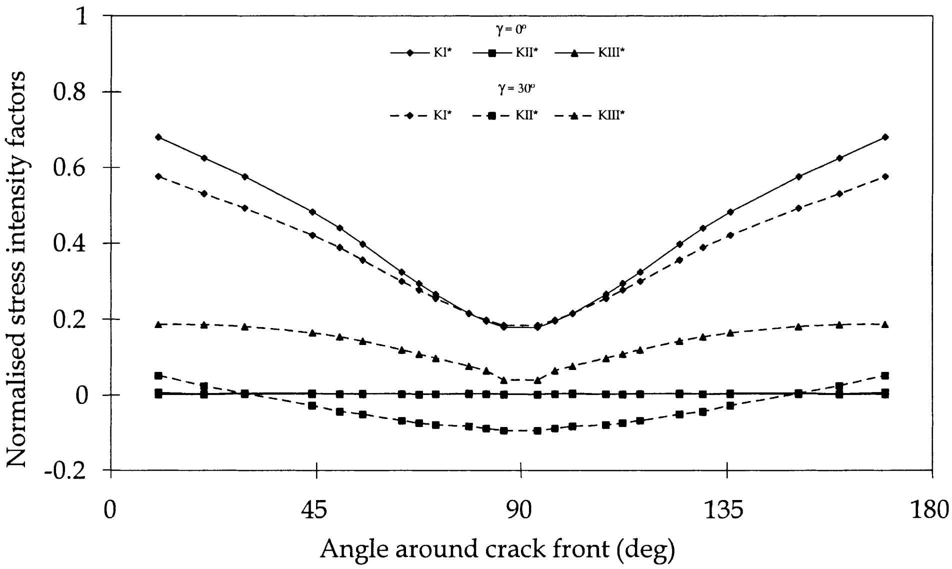

Fig. 10. Stress intensity factors for a semi-elliptical surface crack under thermal gradient.

10. Conclusions

The J-integral for three-dimensional thermoelasticity was presented for analysis of mixed mode thermoelastic crack problems. The main conclusions on the derivation and implementation of the J-integral are as follows.

- It was found that eight subdivisions are necessary to achieve convergence of results.
- The optimum contour radius r was found as $0.3 \leqslant r/a \leqslant 0.7$ for the penny-shaped crack case.
- Special care must be taken when considering a combination of small contour radius and high number of subdivisions, as near singularities which may arise from this combination can affect the values at internal points close to the crack surface, and consequently affect the obtained results.
- The J-integral has been proved to be accurate for stress intensity factor assessment. Values for K_{I} has been found to be within 2.2% difference of analytical values. Less accurate results were obtained for K_{II} and K_{III} values, when compared with analytical results. This discrepancies can occur because even a small error in the temperature values produce large differences in stresses.

Appendix A

In this section, the kernels for the boundary integral equations presented previously, for the calculation of the J-integral are given. The kernels will be obtained in the global coordinate system (x_1, x_2, x_3) illustrated in Fig. 3. Derivatives of kernels differentiated with respect to the global system includes all the components, since they will be necessary for the subsequent transformation into the local coordinate system.

The kernels θ_i^{**} and q_i^{**} in Eq. (43), are obtained by differentiating those in Eq. (42) with respect to x_i, as

$$\theta_i^{**}(\mathbf{X}', \mathbf{x}) = -\frac{r_{,i}}{4\pi\lambda r^2}, \tag{A.1}$$

$$q_i^{**}(\mathbf{X}', \mathbf{x}) = \frac{1}{4\pi r^3}[3r_{,i}r_{,k}n_k - n_i] \tag{A.2}$$

being λ, the thermal conductivity.

The kernels $T_{ij,k}$, $U_{ij,k}$, $\overline{P}_{i,k}$ and $\overline{Q}_{i,k}$ in Eq. (44) are

$$T_{ij,k}(\mathbf{X}', \mathbf{x}) = -\frac{1}{8\pi(1-v)r^3}\left\{(1-2v)\left[3r_{,k}\left(\delta_{ij}\frac{\partial r}{\partial n} + r_{,j}n_i - r_{,i}n_j\right) - \delta_{ij}n_k - \delta_{jk}n_i + \delta_{ik}n_j\right] + 3\frac{\partial r}{\partial n}\left(5r_{,i}r_{,j}r_{,k} - \delta_{ik}r_{,j} - \delta_{jk}r_{,i}\right) - 3n_k r_{,i}r_{,j}\right\}, \tag{A.3}$$

$$U_{ij,k}(\mathbf{X}', \mathbf{x}) = \frac{(1+v)}{8\pi E(1-v)r^2}\left[(3-4v)\delta_{ij}r_{,k} + 3r_{,i}r_{,j}r_{,k} - \delta_{jk}r_{,i} - \delta_{ik}r_{,j}\right], \tag{A.4}$$

$$\overline{P}_{i,k}(\mathbf{X}', \mathbf{x}) = \frac{\alpha(1+v)}{8\pi(1-v)r^2}\left[\frac{\partial r}{\partial n}(\delta_{ik} - 3r_{,i}r_{,k}) + n_k r_{,i} + n_i r_{,k}\right], \tag{A.5}$$

$$\overline{Q}_{i,k}(\mathbf{X}', \mathbf{x}) = \frac{\alpha(1+v)}{8\pi(1-v)r}[r_{,i}r_{,k} - \delta_{ik}]. \tag{A.6}$$

The kernels (A.3)–(A.6) are differentiated again to obtain the kernels $T_{ij,km}$, $U_{ij,km}$, $\overline{P}_{i,km}$ and $\overline{Q}_{i,km}$ of Eq. (45) as

$$\begin{aligned}T_{ij,km}(\mathbf{X}',\mathbf{x}) = &-\frac{3}{8\pi(1-v)r^4}\bigg\{(1-2v)\bigg[5r_{,k}r_{,m}\bigg(\delta_{ij}\frac{\partial r}{\partial n}+r_{,j}n_i-r_{,i}n_j\bigg)\\&+r_{,m}\big(\delta_{ik}n_j-\delta_{jk}n_i-\delta_{ij}n_k\big)+r_{,k}\big(\delta_{im}n_j-\delta_{jm}n_i-\delta_{ij}n_m\big)\\&+\delta_{km}\bigg(r_{,i}n_j-r_{,j}n_i-\delta_{ij}\frac{\partial r}{\partial n}\bigg)\bigg]+\frac{\partial r}{\partial n}\big[35r_{,i}r_{,j}r_{,k}r_{,m}\\&-5r_{,i}\big(\delta_{jm}r_{,k}+\delta_{km}r_{,j}+\delta_{jk}r_{,m}\big)-5r_{,j}(\delta_{im}r_{,k}+\delta_{ik}r_{,m})\\&+\delta_{im}\delta_{jk}+\delta_{ik}\delta_{jm}\big]+n_k\big(\delta_{im}r_{,j}+\delta_{jm}r_{,i}-5r_{,i}r_{,j}r_{,m}\big)\\&+n_m\big(\delta_{ik}r_{,j}+\delta_{jk}r_{,i}-5r_{,i}r_{,j}r_{,k}\big)\bigg\},\end{aligned}\tag{A.7}$$

$$\begin{aligned}U_{ij,km}(\mathbf{X}',\mathbf{x}) = &\frac{(1+v)}{8\pi E(1-v)r^3}\big\{3r_{,k}r_{,m}\big[\delta_{ij}(3-4v)+5r_{,i}r_{,j}\big]\\&-3r_{,j}(\delta_{ik}r_{,m}+\delta_{km}r_{,i}+\delta_{im}r_{,k})-3r_{,i}\big(\delta_{jk}r_{,m}+\delta_{jm}r_{,k}\big)\\&+\delta_{im}\delta_{jk}+\delta_{ik}\delta_{jm}-(3-4v)\delta_{ij}\delta_{km}\big\},\end{aligned}\tag{A.8}$$

$$\begin{aligned}\overline{P}_{i,km}(\mathbf{X}',\mathbf{x}) = &\frac{\alpha(1+v)}{8\pi(1-v)r^3}\bigg[3\frac{\partial r}{\partial n}(\delta_{ik}r_{,m}+\delta_{im}r_{,k}+\delta_{km}r_{,i}-5r_{,i}r_{,k}r_{,m})\\&+n_m(3r_{,i}r_{,k}-\delta_{ik})+n_k(3r_{,i}r_{,m}-\delta_{im})+n_i(3r_{,k}r_{,m}-\delta_{km})\bigg],\end{aligned}\tag{A.9}$$

$$\overline{Q}_{i,km}(\mathbf{X}',\mathbf{x}) = \frac{\alpha(1+v)}{8\pi(1-v)r^2}[3r_{,i}r_{,k}r_{,m}-\delta_{km}r_{,i}-\delta_{im}r_{,k}-\delta_{ik}r_{,m}].\tag{A.10}$$

The kernels $T_{kij,m}$, $U_{kij,m}$, $\overline{P}_{ij,m}$ and $\overline{Q}_{ij,m}$ in Eq. (47) are obtained by differentiating the kernels in Eq. (46) and they are listed below:

$$\begin{aligned}T_{kij,m}(\mathbf{X}',\mathbf{x}) = &\frac{3E}{8\pi(1-v^2)r^4}\bigg\{\frac{\partial r}{\partial n}\big[(1-2v)\delta_{ij}(5r_{,k}r_{,m}-\delta_{km})\\&+v\big[\delta_{ik}\big(5r_{,j}r_{,m}-\delta_{jm}\big)+\delta_{jk}(5r_{,i}r_{,m}-\delta_{im})\big]\\&+5\big[r_{,i}\big(r_{,j}\delta_{km}+r_{,k}\delta_{jm}-7r_{,j}r_{,k}r_{,m}\big)+\delta_{im}r_{,j}r_{,k}\big]\big]\\&+(1-2v)\big[r_{,m}\big(\delta_{ik}n_j+\delta_{jk}n_i\big)+n_k\big(5r_{,i}r_{,j}r_{,m}-\delta_{jm}r_{,i}-\delta_{im}r_{,j}\big)\\&-\delta_{ij}r_{,k}n_m\big]+v\big[5r_{,k}r_{,m}\big(r_{,j}n_i+r_{,i}n_j\big)-n_i\big(\delta_{km}r_{,j}+\delta_{jm}r_{,k}\big)\\&-n_m\big(\delta_{ik}r_{,j}+\delta_{jk}r_{,i}\big)-n_j(\delta_{km}r_{,i}+\delta_{im}r_{,k})\big]\\&+5r_{,i}r_{,j}r_{,k}n_m-(1-4v)\delta_{ij}r_{,m}n_k\bigg\},\end{aligned}\tag{A.11}$$

$$\begin{aligned}U_{kij,m}(\mathbf{X}',\mathbf{x}) = &\frac{1}{8\pi(1-v)r^3}\big\{(1-2v)\big[3r_{,m}\big(\delta_{ik}r_{,j}+\delta_{jk}r_{,i}-\delta_{ij}r_{,k}\big)-\delta_{ik}\delta_{jm}-\delta_{jk}\delta_{im}\\&+\delta_{ij}\delta_{km}\big]+3\big[r_{,i}\big(5r_{,j}r_{,k}r_{,m}-r_{,j}\delta_{km}-r_{,k}\delta_{jm}\big)-\delta_{im}r_{,k}r_{,j}\big]\big\},\end{aligned}\tag{A.12}$$

$$\begin{aligned}\overline{P}_{ij,m}(\mathbf{X}',\mathbf{x}) = &\frac{E\alpha}{8\pi(1-v)r^3}\bigg\{3\frac{\partial r}{\partial n}\bigg[r_{,m}\bigg(\frac{\delta_{ij}}{1-2v}-5r_{,i}r_{,j}\bigg)+\delta_{jm}r_{,i}+\delta_{im}r_{,j}\bigg]+n_m\bigg(3r_{,i}r_{,j}-\frac{\delta_{ij}}{1-2v}\bigg)\\&+n_i\big(3r_{,j}r_{,m}-\delta_{jm}\big)+n_j(3r_{,i}r_{,m}-\delta_{im})\bigg\},\end{aligned}\tag{A.13}$$

$$\overline{Q}_{ij,m}(\mathbf{X}', \mathbf{x}) = \frac{E\alpha}{8\pi(1-\nu)r^2}\left[3r_{,i}r_{,j}r_{,m} - \delta_{jm}r_{,i} - \delta_{im}r_{,j} - \frac{\delta_{ij}}{1-2\nu}r_{,m}\right]. \tag{A.14}$$

Appendix B

In this section, the values at end points are obtained. First, the position of the point within the element must be found in terms of the local intrinsic coordinates (ξ, η). Using the radius of the contour, the position of a point $\mathbf{x}_s$ located just above the element can be calculated. The position $\mathbf{x}_0$ of the nearest point to the element can be obtained from the following equations:

$$\mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial \xi} = 0, \qquad \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial \eta} = 0$$

being $\mathbf{r} = \mathbf{x}_0 - \mathbf{x}_s$. When $\mathbf{r}$ is perpendicular to tangents $\partial\mathbf{r}/\partial\xi$ and $\partial\mathbf{r}/\partial\eta$, the dot product in the above equations is zero and $\mathbf{x}_0$ results the projection of $\mathbf{x}_s$ on the element. Because of the non-linearity of the above equations, they can be solved with a Newton–Raphson scheme leading to the intrinsic coordinates ξ_0 and η_0 of the nearest point on the element. The initial solution for the iterative scheme is set to the intrinsic coordinates of the nearest node to $\mathbf{x}_s$.

Once the point $\mathbf{x}_0$ is located on the element, the shape functions M^α $(\alpha = 1, 8)$ and shape functions derivatives $\partial M^\alpha/\partial\xi$ and $\partial M^\alpha/\partial\eta$ are found for the intrinsic coordinates ξ_0, η_0. Since several derivatives and second derivative values are required, the way of obtaining them will be explained generically for a function f and then particularised in each case.

By virtue of the chain differentiation rule, the following can be written:

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial \xi}, \qquad \frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial \eta},$$

where $j = 1, 3$. As the element is on the crack plane, its normal is parallel to axis x_2, as a consequence $\partial x_2/\partial\xi = \partial x_2/\partial\eta = 0$ and the above relation reduces to

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_3}{\partial \xi} \\ \frac{\partial x_1}{\partial \eta} & \frac{\partial x_3}{\partial \eta} \end{Bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_3} \end{Bmatrix}$$

from which the derivatives in the local crack coordinate system can be calculated as

$$\begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_3} \end{Bmatrix} = \frac{1}{\frac{\partial x_1}{\partial \xi}\frac{\partial x_3}{\partial \eta} - \frac{\partial x_1}{\partial \eta}\frac{\partial x_3}{\partial \xi}} \begin{Bmatrix} \frac{\partial x_3}{\partial \eta} & -\frac{\partial x_3}{\partial \xi} \\ -\frac{\partial x_1}{\partial \eta} & \frac{\partial x_1}{\partial \xi} \end{Bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}, \tag{B.1}$$

where the derivatives respect to the intrinsic values are evaluated as

$$\frac{\partial f}{\partial \xi} = \sum_{\alpha=1}^{8} \frac{\partial M^\alpha}{\partial \xi} f^\alpha, \qquad \frac{\partial f}{\partial \eta} = \sum_{\alpha=1}^{8} \frac{\partial M^\alpha}{\partial \eta} f^\alpha.$$

So, the required derivatives can be found by adequately setting the function to the required thermoelastic variable.

The stresses are calculated from the values of tractions and tangential stresses at boundary points as given by Aliabadi and Rooke (1991). In this work, this procedure is extended to thermoelasticity. A local coordinate system is defined such that e^0_{ij} are components (directional cosines) of the unit vectors of the orthogonal system of axes defining a coordinate system x^0 as shown in Fig. 11.

If u^0_i, ϵ^0_{ij}, σ^0_{ij} and t^0_i are the displacements, strains, stresses and tractions, respectively, in the local system of coordinates and θ are the temperatures, the stress components σ^0_{3i} can be expressed as

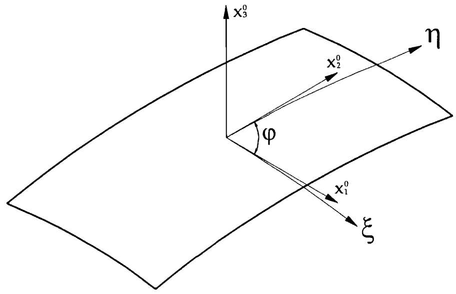

Fig. 11. Orthogonal coordinate system at the surface.

$$\sigma_{3i}^0 = t_i^0, \quad i = 1, 3,$$

and from Hooke's law for thermoelasticity

$$\sigma_{ij} = 2\mu\left[\epsilon_{ij} + \frac{v}{1-2v}\epsilon_{kk}\delta_{ij} - \frac{(1+v)}{1-2v}\alpha\theta\delta_{ij}\right] \tag{B.2}$$

making $\sigma_{33}^0 = t_3^0$ yields

$$\epsilon_{33}^0 = \frac{1}{1-v}\left[\frac{1-2v}{2\mu}t_3^0 - v\left(\epsilon_{11}^0 + \epsilon_{22}^0\right) + (1+v)\alpha\theta\right].$$

Eliminating ϵ_{33}^0 from Eq. (B.2) leads to

$$\begin{aligned}
\sigma_{11}^0 &= \frac{1}{1-v}\left\{vt_3^0 + 2\mu\left[\epsilon_{11}^0 + v\epsilon_{22}^0 - (1+v)\alpha\theta\right]\right\},\\
\sigma_{22}^0 &= \frac{1}{1-v}\left\{vt_3^0 + 2\mu\left[\epsilon_{22}^0 + v\epsilon_{11}^0 - (1+v)\alpha\theta\right]\right\},\\
\sigma_{12}^0 &= 2\mu\epsilon_{12}^0.
\end{aligned} \tag{B.3}$$

The displacements and tractions, are related to the local coordinate system via the transformation matrix e_{ij}^0, as follows:

$$u_i^0 = e_{ij}^0 u_j, \quad t_i^0 = e_{ij}^0 t_j.$$

The surface representation using shape functions is

$$\mathbf{x} = \sum_{\alpha=1}^{m} M^\alpha(\xi,\eta)\mathbf{x}^\alpha$$

from which the two tangent vectors at a surface point (ξ_0, η_0) are given by

$$\mathbf{h}(\xi_0,\eta_0) = \left.\frac{\partial\mathbf{x}}{\partial\xi}\right|_{\substack{\xi=\xi_0\\ \eta=\eta_0}} = \sum_{\alpha=1}^{m}\left.\frac{\partial M^\alpha(\xi,\eta)}{\partial\xi}\right|_{\substack{\xi=\xi_0\\ \eta=\eta_0}}\mathbf{x}^\alpha, \quad \mathbf{g}(\xi_0,\eta_0) = \left.\frac{\partial\mathbf{x}}{\partial\eta}\right|_{\substack{\xi=\xi_0\\ \eta=\eta_0}} = \sum_{\alpha=1}^{m}\left.\frac{\partial M^\alpha(\xi,\eta)}{\partial\eta}\right|_{\substack{\xi=\xi_0\\ \eta=\eta_0}}\mathbf{x}^\alpha,$$

and the outward normal vector as the vector product of these two vectors, as

$$\mathbf{d}(\xi_0,\eta_0) = \mathbf{h}(\xi_0,\eta_0) \times \mathbf{g}(\xi_0,\eta_0).$$

The local orthogonal unit vectors are defined by

$$e^0_{1i} = \frac{h_i(\xi_0,\eta_0)}{|\mathbf{h}(\xi_0,\eta_0)|}, \qquad e^0_{2i} = \frac{g_i(\xi_0,\eta_0)}{|\mathbf{g}(\xi_0,\eta_0)|},$$

$$e^0_{3i} = \frac{1}{|\mathbf{d}(\xi_0,\eta_0)|}\left[|\mathbf{h}(\xi_0,\eta_0)|g_i(\xi_0,\eta_0) - h_j(\xi_0,\eta_0)g_j(\xi_0,\eta_0)\frac{h_i(\xi_0,\eta_0)}{|\mathbf{h}(\xi_0,\eta_0)|}\right],$$

where $|\mathbf{h}(\xi_0,\eta_0)| = \sqrt{h_i h_i}$, $|\mathbf{g}(\xi_0,\eta_0)| = \sqrt{g_i g_i}$ and $|\mathbf{d}(\xi_0,\eta_0)| = \sqrt{d_i d_i}$. From Fig. 11, it is possible to relate the intrinsic coordinates (ξ,η) to the surface tangential directions $(x^0_1,\ x^0_2)$ as

$$\xi = \frac{1}{|\mathbf{h}(\xi_0,\eta_0)|}\left[x^0_1 - x^0_2 \tan^{-1}\varphi\right], \qquad \eta = \frac{1}{|\mathbf{g}(\xi_0,\eta_0)|}\left[x^0_2 \sin^{-1}\varphi\right].$$

Differentiating ξ and η with respect to x^0_1, and x^0_2 yield

$$\frac{\partial\xi}{\partial x^0_1} = \frac{1}{|\mathbf{h}(\xi_0,\eta_0)|}, \qquad \frac{\partial\eta}{\partial x^0_1} = 0 \qquad \frac{\partial\xi}{\partial x^0_2} = \frac{1}{|\mathbf{h}(\xi_0,\eta_0)|}\tan^{-1}\varphi, \qquad \frac{\partial\eta}{\partial x^0_2} = \frac{1}{|\mathbf{g}(\xi_0,\eta_0)|}\sin^{-1}\varphi.$$

The strain tensor in the local coordinate system and the temperature at point (ξ_0,η_0) can be evaluated as

$$\epsilon^0_{ij} = \frac{1}{2}\left(\frac{\partial u^0_i}{\partial x^0_j} + \frac{\partial u^0_j}{\partial x^0_i}\right) = \frac{1}{2}\left(\frac{\partial u^0_i}{\partial\xi}\frac{\partial\xi}{\partial x^0_j} + \frac{\partial u^0_i}{\partial\eta}\frac{\partial\eta}{\partial x^0_j} + \frac{\partial u^0_j}{\partial\xi}\frac{\partial\xi}{\partial x^0_i} + \frac{\partial u^0_j}{\partial\eta}\frac{\partial\eta}{\partial x^0_i}\right), \tag{B.4}$$

$$\theta = \sum_{\alpha=1}^{m} M^\alpha(\xi_0,\eta_0)\theta^\alpha, \tag{B.5}$$

where the derivatives with respect to the intrinsic coordinates (ξ,η) are

$$\frac{\partial u^0_i}{\partial\xi} = \sum_{\alpha=1}^{m}\frac{\partial M^\alpha(\xi,\eta)}{\partial\xi}u^{0,\alpha}_i, \qquad \frac{\partial u^0_i}{\partial\eta} = \sum_{\alpha=1}^{m}\frac{\partial M^\alpha(\xi,\eta)}{\partial\eta}u^{0,\alpha}_i.$$

Substituting Eqs. (B.4) and (B.5) into Eq. (B.3) leads to the stress tensor in the local coordinate system. Finally, the stress tensor in the global coordinate system is obtained from the transformation,

$$\sigma_{ij} = e^0_{ki}e^0_{ni}\sigma^0_{kn}.$$

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